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# A THEOREM IN DIFFERENCE EQUATIONS ON THE ALTERNATION OF NODES OF LINEARLY INDEPENDENT SOLUTIONS.

BY E. J. MOULTON.

Consider the difference equation

$$(1) \quad L(i)u(i+1) + M(i)u(i) + N(i)u(i-1) = 0 \quad (i = 1, 2, \dots, n).$$

Plot a solution  $u(i)$  of this equation on an  $X$ -axis, making  $u(i)$  correspond to a point  $x_i$ , where  $x_i < x_{i+1}$ , and joining successive points by straight line segments; a point  $x$  at which this broken line meets the axis will be called a *node* of the solution. Concerning the nodes of linearly independent solutions, the following analogon of a classic theorem of Sturm for differential equations may be stated:

**THEOREM.** *For any linear homogeneous difference equation of the form (1), where  $L(i)N(i) > 0$  for every value of  $i$  ( $i = 1, 2, \dots, n$ ), the nodes of any two linearly independent solutions separate each other.\**

The proof of this theorem can be effected as follows. Let  $u_1(i)$  and  $u_2(i)$  be any two linearly independent solutions. Then

1°. No node of  $u_1(i)$  coincides with a node of  $u_2(i)$ . For if  $x$  were a node of both  $u_1(i)$  and  $u_2(i)$  it would be a node of every solution; but supposing  $x_k \leq x \leq x_{k+1}$ , it is obvious that the solution determined by the conditions  $u(k) = 1, u(k+1) = 1$  has no node at  $x$ .

Next, using the notation

$$(2) \quad W(k) \equiv u_1(k+1)u_2(k) - u_1(k)u_2(k+1) \quad (k = 0, 1, \dots, n),$$

it is known from the general theory of difference equations that  $W(k)$  is distinct from zero; and one may prove

2°. For any values of  $k$  and  $l$  such that  $W(k)$  and  $W(k+l)$  are defined,

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\* The proof of this theorem was given for an exercise by Professor Bôcher in a course in differential equations at Harvard; and proofs more or less similar to the one here given were worked out simultaneously with and independently of this one by Messrs. Allen, Brand, Fort and Graustein. A theorem very similar to this one is given by Porter in these *Annals*, 2 Series, vol. 3, 1901-1902, on p. 65. He makes the assumption that neither of the solutions considered has a node at  $x_0$ . And his proof, which is very different in character from the one here given, depending upon the introduction of an auxiliary parameter, seems to be lacking in completeness; it does not seem obvious, for example, that the  $v$ -points (i. e., nodes) all vary in one direction as the parameter  $z$  increases.

$$\frac{W(k)}{W(k+l)} > 0.$$

It is no restriction to assume  $l \geq 0$ ; and for  $l = 0$  the result is immediate. From the identities

$$L(k+1)u_1(k+2) + M(k+1)u_1(k+1) + N(k+1)u_1(k) = 0,$$

$$L(k+1)u_2(k+2) + M(k+1)u_2(k+1) + N(k+1)u_2(k) = 0,$$

one easily obtains

$$L(k+1)W(k+1) - N(k+1)W(k) = 0.$$

Hence, since  $W(k+1) \neq 0$  and  $L(k+1)N(k+1) > 0$ ,

$$\frac{W(k)}{W(k+1)} = \frac{L(k+1)}{N(k+1)} > 0;$$

and therefore

$$\frac{W(k)}{W(k+l)} = \prod_{j=1}^l \frac{L(k+j)}{N(k+j)} > 0.$$

The theorem is without content unless at least one of the solutions has two or more nodes on the interval  $x_0 \leq x \leq x_{n+1}$ . Suppose then that  $u_1(i)$  has two successive nodes at  $x'$  and  $x''$ :

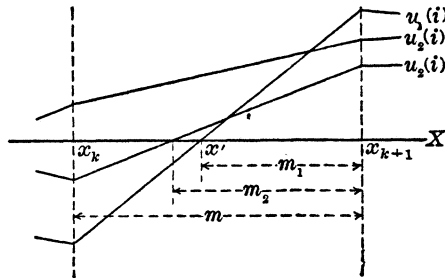
$$x_k \leq x' < x_{k+1}, \quad x_{k+l} < x'' \leq x_{k+l+1}, \quad l \geq 1.$$

One may prove next

3°. If  $u_2(i)$  has no node on the interval  $x' < x < x''$  then

$$(3) \quad a) \frac{W(k)}{u_1(k+1)u_2(k+1)} > 0; \quad b) \frac{W(k+l)}{u_1(k+l)u_2(k+l)} < 0.$$

Consider the interval  $x_k \leq x \leq x_{k+1}$ ; the possibilities for  $u_2(i)$  are indicated in the figure; it is to be observed that if the signs of  $u_1(i)$  or of  $u_2(i)$



or of both  $u_1(i)$  and  $u_2(i)$  are changed the following argument is not altered. Now  $u_1(k+1)$  and  $u_2(k+1)$  are not zero and hence from (2)

$$\frac{W(k)}{u_1(k+1)u_2(k+1)} = \frac{u_2(k)}{u_2(k+1)} - \frac{u_1(k)}{u_1(k+1)}.$$

If  $u_1(k) = 0$ , then, using  $1^\circ$ ,

$$\frac{u_2(k)}{u_2(k+1)} > 0;$$

hence (3a). If  $u_1(k) \neq 0$ , then either

$$\frac{u_2(k)}{u_2(k+1)} \geq 0, \quad \frac{-u_1(k)}{u_1(k+1)} > 0,$$

and hence (3a); or, referring to the figure, and using  $1^\circ$ ,

$$\frac{u_2(k)}{u_2(k+1)} - \frac{u_1(k)}{u_1(k+1)} = \left(-\frac{m-m_2}{m_2}\right) - \left(-\frac{m-m_1}{m_1}\right) = -m \frac{m_1-m_2}{m_1m_2} > 0.$$

Therefore (3a) holds in all cases. (3b) can be established in a similar manner.

Now suppose that  $u_2(i)$  has no node on the interval  $x' < x < x''$ ; then dividing (3a) by (3b),

$$\frac{W(k)}{W(k+l)} \cdot \frac{u_1(k+l)}{u_1(k+1)} \cdot \frac{u_2(k+l)}{u_2(k+1)} < 0.$$

Hence by  $2^\circ$  and the hypothesis that  $u_1(i)$  has no node on the interval  $x' < x < x''$ , it follows that

$$\frac{u_2(k+l)}{u_2(k+1)} < 0.$$

If  $l = 1$  this leads at once to a contradiction; if  $l > 1$  this implies that  $u_2(i)$  has a node on  $x_{k+1} \leq x \leq x_{k+l}$ , and one is again led to a contradiction. Hence between every pair of nodes of one solution there is a node of the other; that is, the nodes separate each other.

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